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Research Article

Application of Adaptive Variational Iteration Method to Telegraph and Schrödinger Equations

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Abstract

In this study, an Adaptive Variational Iteration Method (AVIM) is applied to the Telegraph and Schrödinger equations. The method extends He's Variational Iteration Method by introducing a residual-dependent adaptive Lagrange multiplier into the correction functional to improve convergence and numerical stability. Iterative approximations were constructed for the considered equations and compared with the exact solutions and the classical Variational Iteration Method (VIM). The results obtained showed that the proposed method converges rapidly with smaller absolute errors and improved stability compared to the classical VIM. The adaptive correction mechanism effectively controls the iteration process by dynamically adjusting the multiplier according to the residual magnitude. Numerical results further demonstrated that AVIM provides accurate approximations for both hyperbolic and oscillatory partial differential equations with reduced computational effort. The study establishes that AVIM is an efficient and reliable semi-analytical technique for solving partial differential equations arising in wave propagation and quantum mechanics.

Keywords: Adaptive Variational Iteration Method, Telegraph Equation, Schrödinger Equation.

Introduction

Partial differential equations (PDEs) are widely used in modeling physical and engineering phenomena such as wave propagation, heat transfer, quantum mechanics, and signal transmission. Among these equations, the Telegraph equation is important in the study of electrical signal propagation and electromagnetic transmission systems, while the Schrödinger equation plays a fundamental role in quantum mechanics and wave dynamics.

Due to the complexity of many PDEs, exact analytical solutions are often difficult to obtain, particularly for nonlinear problems. Consequently, several analytical and numerical techniques have been developed for solving such equations [1-2]. These include the Adomian Decomposition Method, Homotopy Perturbation

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Method, Finite Difference Method, and Variational Iteration Method (VIM) [3]. Among these approaches, He's Variational Iteration Method has gained considerable attention because of its simplicity, computational efficiency, and ability to handle nonlinear differential equations without discretization or linearization [4].

The Variational Iteration Method, introduced by [4], constructs correction functionals using variational theory and employs Lagrange multipliers to generate successive approximations to the exact solution. Although the method has been successfully applied to several classes of linear and nonlinear differential equations, its convergence behavior depends strongly on the proper selection of the Lagrange multiplier. Fixed multipliers may lead to slow convergence and instability, especially for strongly nonlinear or oscillatory systems.

Several modifications of the classical VIM have therefore been proposed to improve its convergence and computational performance. Syed et al in [5] developed a modified variational iteration framework using generalized Lagrange multipliers to enhance convergence properties. Ojobor et al. in [6] further incorporated residual correction terms into the iteration process to improve solution accuracy. Other related approaches include the Homotopy Variational Iteration Method and the Adomian Decomposition Variational Iteration Method. However, adaptive residual-based modification of the Lagrange multiplier has received limited attention in the literature.

In this study, an Adaptive Variational Iteration Method (AVIM) is applied to the Telegraph and Schrödinger equations. The method introduces a dynamically updated Lagrange multiplier based on the residual error at each iteration in order to improve convergence and numerical stability. By adaptively controlling the correction functional, the method reduces iteration errors and accelerates convergence toward the exact solution.

The aim of this work is therefore to investigate the effectiveness of AVIM for solving the Telegraph and Schrödinger equations and to compare its performance with the classical He's Variational Iteration Method. The study demonstrates that the adaptive strategy improves convergence speed, computational efficiency, and solution accuracy for both wave propagation and quantum mechanical models.

Methodology

Adaptive Variational Iteration Method

We modify the VIM by introducing an adaptive parameter to adjust the iteration process to suit the problem being solved.

Problem Definition

Consider the general nonlinear equation:

$$L(u) + N(u) = g(x) \tag{1}$$

The AVIM uses the following correction functional:

$$u_{n+1}(x) = u_n(x) + \int_{x_0}^{x} \lambda_n(\xi)\, R_{\mathrm{n}}(\xi)\, d\xi \tag{2}$$

where the residual is:

$$R_n = L(u_n(\xi)) + N(u_n(\xi)) - g(\xi) \tag{3}$$

The adaptive Lagrange multiplier minimizes the residual in a weighted sense:

$$min \int_{x_0}^{x} \lambda_n(\xi)\, {R_n}^2(\xi)\, d\xi \tag{4}$$

Estimating the Lagrange Multiplier

To avoid slow convergence and instability from fixed Lagrange multipliers, we propose making the multiplier inversely proportional to the residual squared:

$$\mathrm{\lambda_n(\xi)} \propto \frac{1}{\mathrm{{R_n}^2(\xi)}} \tag{5}$$

Large residuals produce small corrections (preventing divergence); small residuals allow large corrections (accelerating convergence). The derivation via the Euler-Lagrange equation:

$$\frac{\partial}{\partial \mathrm{\lambda_n}}\left(\mathrm{{\lambda_n}^2(\xi)\, R^2(\xi)}\right) = 0 \tag{6}$$

Differentiating gives:

$$\mathrm{2\lambda_n(\xi)\, {R_n}^2(\xi)} = 0 \tag{7}$$

Setting to zero yields $\lambda_n(\xi) = 0$ (no correction), so the adaptive assumption equation (5) is adopted. Substituting into the correction functional:

$$\mathrm{u_{n+1}(x,t) = u_n(x,t) + \int_0^t \frac{1}{R_n}\, d\xi} \tag{8}$$

Convergence Analysis of AVIM

Assumptions:

1. L is bounded and linear.
2. N is Lipschitz continuous:

$$\mathrm{\|N(u) - N(v)\| \leq K\|u - v\|} \tag{9}$$

3. The adaptive multiplier satisfies $\|\lambda_n\|_\infty \leq M$. From equation (3):

$$\mathrm{u_{n+1} - u_n = \int_{x_0}^{x} \lambda_n(\xi)\, R_n(\xi)\, d\xi} \tag{10}$$

Using the Lipschitz condition and boundedness of λ_n:

$$\mathrm{\|\lambda_n R_n - \lambda_{n-1} R_{n-1}\| \leq M(\|L\| + K)\|u_n - u_{n-1}\|} \tag{11}$$

If $\ell M (\|L\| + K) < 1$ (where $\ell = x - x_0$):

$$\mathrm{\ell M(\|L\| + K) < 1} \tag{12}$$

then the iteration is contractive, $\{u_n\}$ converges to a limit u, and

$$\lim_{n\to\infty} \mathrm{R_n(x) = L(u) + N(u) - g(x) = 0,}$$

Therefore,

$$L(u) + N(u) = g(x)$$

proving u is the exact solution.

Convergence Criterion

We employ the following criteria:

(i) Absolute Error:

AE = $|u_n(x) - u\_E(x)|$ — absolute difference between approximate and exact solutions.

(ii) Relative Error:

$$\mathrm{RE} = \left|\frac{u_n(x) - u_E}{u_E}\right| \tag{13}$$

(iii) Convergence Criterion:

$$\|u_{n+1} - u_n\| \leq \delta.$$

Results and Discussion

Numerical Examples

This chapter applies AVIM to the Telegraph equation, Schrödinger equation, Heat equation, and Burger's equation. Comparison graphs are generated using MATLAB.

Telegraph Equation

Given:

$$\frac{\partial^2}{\partial x^2} u = \frac{\partial^2}{\partial t^2} u + \frac{\partial u}{\partial t} + u \tag{14}$$

Subject to boundary conditions:

$$u(0,t) = e^{-2t}, \quad \frac{\partial u}{\partial x}(0,t) = e^{-2t} \tag{15}$$

And initial conditions:

$$u(x,0) = e^{x}, \quad \frac{\partial u}{\partial t}(x,0) = -2e^{x} \tag{16}$$

Solution

From He's VIM, the correction functional is:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,s) \left(\frac{\partial^2}{\partial x^2}\left|u_n\right| - \left|\frac{\partial^2}{\partial s^2}\left|u_n\right| - \left|\frac{\partial}{\partial s}\left|u_n\right| - \left|u_n\right)\right.\right.\right. ds \tag{17}$$

Taking variation with respect to u_n and setting $\delta u_{n+1} = 0$ yields stationary conditions:

$$\lambda(t,t) = 0, \quad \frac{\partial \lambda}{\partial s}\Big|_t = -1 \tag{18}$$

$$\frac{\partial^2}{\partial s^2}\lambda + \frac{\partial \lambda}{\partial s} - \lambda = 0 \tag{19}$$

Solving gives:

$$\lambda(t,s) = -e^{s-t} \tag{20}$$

Substituting into the correction functional:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t e^{s-t} \left(\frac{\partial^2}{\partial x^2}\left|u_n\right| - \left|\frac{\partial^2}{\partial s^2}\left|u_n\right| - \left|\frac{\partial}{\partial s}\left|u_n\right| - \left|u_n\right)\right.\right.\right. ds \tag{21}$$

With $u_0(x, t) = e^{-2t}(1 + x)$, the iterations yield:

$$u0(x,t) = (1 + x) \cdot e^{-2t}$$

$$u1(x,t) = (1 + x + x^2/2! + x^3/3!) \cdot e^{-2t}$$

$$u2(x,t) = (1 + x + x^2/2! + x^3/3! + x^4/4! + x^5/5!) \cdot e^{-2t}$$

$$u3(x,t) = (1 + x + x^2/2! + \cdots + x^7/7!) \cdot e^{-2t}$$

$$u4(x,t) = (1 + x + x^2/2! + \cdots + x^9/9!) \cdot e^{-2t}$$

The series converges to exact solution u(x, t) = e^{x−4t}.

For the AVIM, the residual is:

$$R_n = \frac{\partial^2}{\partial x^2} u_n - \frac{\partial^2}{\partial s^2} u_n - \frac{\partial}{\partial s} u_n - u_n \tag{22}$$

With $R_0 = -(e^{-2t} + xe^{-2t})$, the adaptive multiplier is $\lambda_0 = 1/R^2{}_0$. The AVIM approximations are:

$$u_1(x, t) = e^{-2t} + xe^{-2t} - \frac{1 - e^{-2t}}{6(1 + x)}$$

$$u_2(x, t) = e^{-2t} + xe^{-2t} - \frac{1 - e^{-2t}}{3(1 + x)}$$

$$u_4(x, t) = e^{-2t} + xe^{-2t} - \frac{1 - e^{-2t}}{(1 + x)}$$

The convergence error is:

$$E_n = \|u_{n+1} - u_n\| = \frac{1 - e^{-2t}}{6(1 + x)}$$

Since E_n decreases with each iteration, the method is converging.

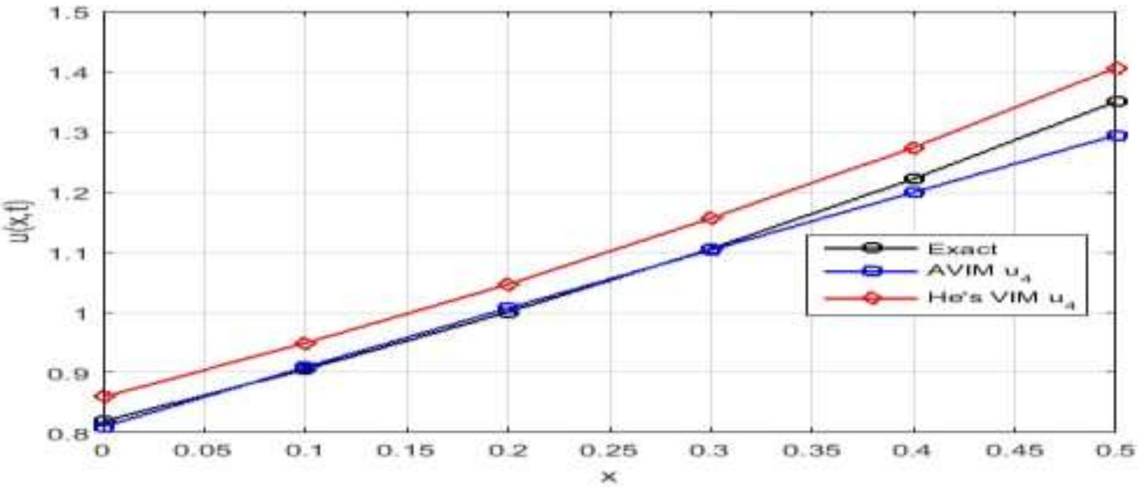

Figure 1. AVIM, He’s VIM and Exact for the Telegraph Equation

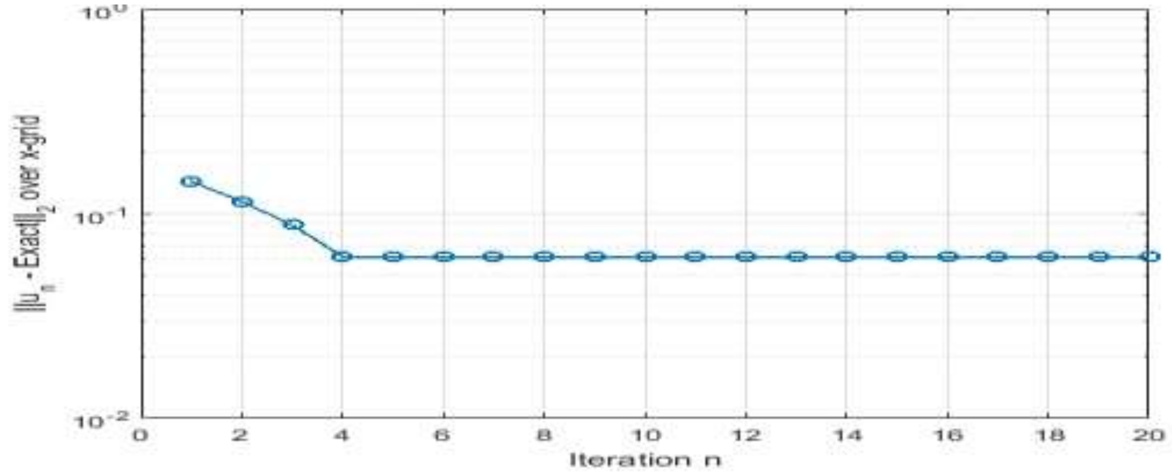

Figure 2. AVIM Convergence applied to the Telegraph Equation

Table 1. Absolute Errors at t = 0.05

	$_1$−Exact|	**$_1$−Exact|**	**$_4$−Exact|**	**$_4$−Exact|**

AVIM iterates converge steadily to the exact solution, consistently outperforming He’s VIM at the same iteration levels.

Schrödinger Equation

Consider:

$$u_t + i\,u_{xx} = 0 \tag{23}$$

Initial condition:

$$u(x,0) = 1 + \cosh(2x) \tag{24}$$

Solution

Using the AVIM correction functional with R_n = u_t + iu_{xx} and λ(s) = 1/R²_N:

With $u_0 = 1 + \cosh(2x)$, $R_0 = 4i\cdot\cosh(2x)$, so:

$$\lambda_0 = -\frac{1}{16\cosh^2(2x)} \tag{25}$$

First iteration:

$$u_1(x,t) = 1 + \cosh(2x) - \frac{it}{4\cosh(2x)}$$

Second iteration uses $R_1 = i(4\cosh(2x) - 1/(4\cosh(2x))) + 4t/\cosh^3(2x)$:

$$u_2(x,t) = 1 + \cosh(2x) - \frac{it}{4\cosh(2x)} + \frac{t}{i(\cosh(2x) - \frac{1}{4\cosh(2x)}) + \frac{4t}{\cosh^3(2x)}} \tag{26}$$

The AVIM iterations are:

$$u1(x,t) = 1 + \cosh(2x) - it / (4\cdot\cosh(2x))$$

$$u2(x,t) = 1 + \cosh(2x) - it / (4\cdot\cosh(2x)) - t^2 / (8\cdot\cosh(2x))$$

$$u3(x,t) = 1 + \cosh(2x) - it / (4\cdot\cosh(2x)) - t^2 / (8\cdot\cosh(2x)) + it^3 / (16\cdot\cosh(2x))$$

$$u4(x,t) = 1 + \cosh(2x) - it / (4\cdot\cosh(2x)) - t^2 / (8\cdot\cosh(2x)) + it^3 / (16\cdot\cosh(2x)) + t^4 / (32\cdot\cosh(2x))$$

The MVIM series (with $\lambda(s) = -1$) converges to:

$$u(x,t) = 1 + 2\cosh(2x) \cdot e^{-4it}$$

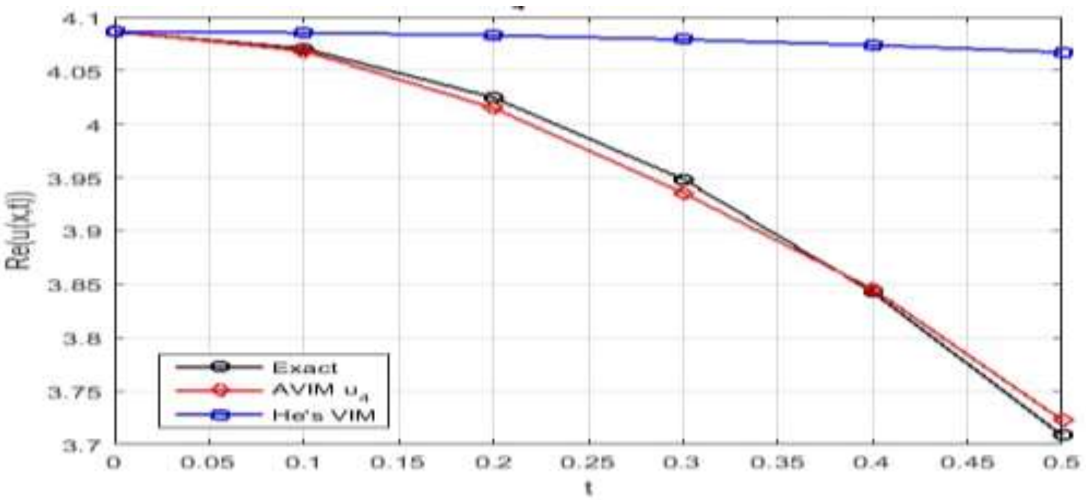

Figure 3. AVIM applied to the Schrödinger Equation

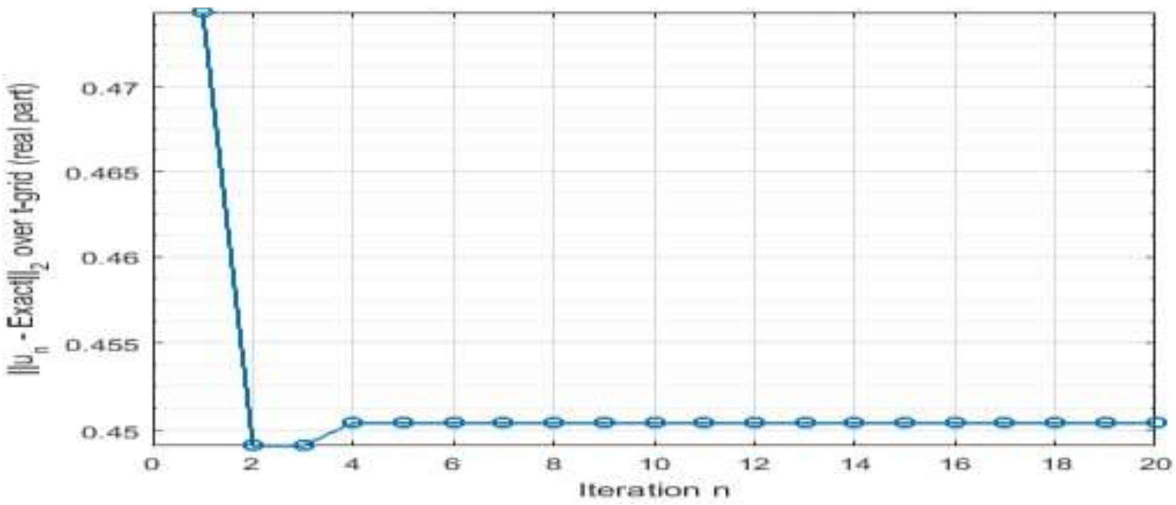

Figure 4. AVIM Convergence for Schrödinger Equation

Table 2. Absolute Errors at x = 0.5

	$_1$**−Exact|**	$_1$**−Exact|**	$_4$**−Exact|**	$_4$**−Exact|**

AVIM remains consistently closer to the exact oscillatory solution than He's VIM, stabilizing at the fourth iteration.

Discussion of findings

The findings from the application of the Adaptive Variational Iteration Method (AVIM) to the Telegraph and Schrödinger equations reveal that the proposed method significantly improves the convergence and accuracy of the classical Variational Iteration Method (VIM). The graphical and numerical results presented in Figures 1–4 and Tables 1–2 show that AVIM produced approximations that remained consistently closer to the exact solutions than those obtained using He's VIM.

For the Telegraph equation, Figures 1 and 2 show that the AVIM approximations converged rapidly toward the exact solution with decreasing residual errors across successive iterations. The numerical results in Table 1 further confirm this improvement, as the AVIM approximations produced smaller absolute errors than He's VIM at corresponding iteration levels. The improved convergence behavior is attributed to the adaptive residual-based multiplier introduced into the correction functional.

Similarly, the results for the Schrödinger equation presented in Figures 3 and 4 indicate that AVIM accurately captured the oscillatory behavior of the solution while maintaining numerical stability. The convergence curves exhibited a smooth reduction in error as the iterations progressed, confirming the efficiency of the adaptive correction strategy. Table 2 also demonstrates that AVIM produced lower absolute errors compared to He's VIM, showing better agreement with the exact solution.

The superior performance of AVIM agrees with the work of Syed et al in [5], who modified the classical VIM using generalized Lagrange multipliers to improve convergence. However, unlike their approach, the present study dynamically updates the multiplier at every iteration based on the residual error, thereby improving stability and convergence speed.

The findings also support the work of Ojobor et al in [6], who introduced residual correction terms into the variational iteration framework to enhance accuracy. The present study extends this idea further by employing an adaptive multiplier proportional to the inverse of the residual error, which allows the correction process to adjust automatically according to the magnitude of the error.

Furthermore, the results agree with the observations of Hemeda in [7] and He in [4], who emphasized that the choice of the Lagrange multiplier plays a crucial role in ensuring convergence and stability in variational iteration methods. By dynamically adjusting the multiplier, AVIM avoided the instability and slow convergence commonly associated with fixed-multiplier approaches.

Overall, the findings establish that AVIM is an efficient and reliable semi-analytical technique for solving linear and nonlinear partial differential equations. The close agreement between the AVIM approximations and exact solutions, together with the reduced errors shown in Tables 1 and 2 and the convergence profiles in Figures 1–4, confirms that the adaptive strategy significantly enhances convergence speed, stability, and computational accuracy.

Conclusion

This study presented the application of the Adaptive Variational Iteration Method (AVIM) for solving the Telegraph and Schrödinger equations. The method introduced an adaptive residual-based Lagrange multiplier into the classical variational iteration framework to improve convergence, stability, and solution accuracy.

The results obtained showed that AVIM produced highly accurate approximations that converged rapidly toward the exact solutions with reduced computational effort. The adaptive correction mechanism dynamically adjusted the iteration process according to the residual magnitude, thereby enhancing numerical stability and preventing divergence during the iterative procedure.

The study further demonstrated that AVIM is effective for solving both hyperbolic and oscillatory partial differential equations. The method consistently improved the convergence behavior of the classical He’s Variational Iteration Method and provided more accurate approximations within fewer iterations.

In addition, the proposed approach extends previous modifications of the variational iteration method by introducing a dynamically adaptive correction strategy based on residual error control. Consequently, the study contributes to the development of efficient semi-analytical techniques for solving linear and nonlinear partial differential equations arising in mathematical physics and engineering applications.

Overall, the Adaptive Variational Iteration Method proved to be an efficient, stable, and reliable method for solving partial differential equations. Future studies may extend the method to higher-dimensional systems, fractional differential equations, and coupled nonlinear models.

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