Approximate Solution of Bratu Differential Equation Using Falker-Type Method

Nicholas S Yakusak[1] and Esther O. Taiwo[2]

[1]Department of Mathematics, Federal College of Education, Okene, Nigeria.
[2]Department of Mathematical Science, Ajayi Crowther University, Oyo, Nigeria.

Received 3 November 2023; Acceptance 15 December 2023; Published 20 January 2024

Abstract
In this paper, approximate solution of Bratu Differential Equation (BDEs) using Falker-type method for solving second order BDEs is presented. The approach of collocation and interpolation technique is adopted to derive the new method, which is implemented in block mode to get approximation at grids points simultaneously. The method is of order five (5), zero-stable, consistent and convergent with good region of absolute stability. The tabular and graphical presentations of the numerical results to the problems considered, demonstrate the effectiveness of the scheme in comparison with exact equation. The method is therefore recommended for solving second order BDEs.
Keyword: *Bratu-type Equation, Falker-type Method, Differential Equation*

Introduction

Real life situations concerned with the rate of change of one quantity with respect to another give rise to equation (1). The standard Bratu problem of equation (1) is applicable in areas such as: The fuel ignition model of the theory of thermal combustion, the thermal reaction process model, the Chandrasekhar model of expansion of Universe, radiated heat transfer, nanotechnology and theory of chemical reaction [1].
Consider the second Order Differential Equations (ODEs) of the form

$$y''(x) + \lambda e^{\mu(x)} = 0, \quad 0 \leq x \leq 1 \tag{1}$$

Subject to the Initial Value Problem (IVPs)

$$y(0) = \alpha, \; y'(0) = \gamma \tag{2}$$

Where α, γ, λconstant number for are $y(x)$ is the unknown functions.
The Bratu IVPs have been studied extensively by many researches. [3] Studied a numerical solution of (1) using variational iterative method. [5] Considered Bratu's problem by the means of modified Homotopy perturbation method. [8] Applied Adomian decomposition method to study the Bratu-type equation. [4] Developed an Algorithm using Runge-Kutta method of order four and five for first order system of Ordinary Differential Equations (ODEs). [7] Investigate numerical solutions of second order IVPs of Bratu-type equation using sixth order Runge- Kutta seven stages method. [2] Proposed a method for finding an approximate function for Bratu differential equations (BDEs), in which trigonometric basic functions. [6] Proposed a new approach for solving one-dimensional Bratu's problem which depends on Bernstein polynomial approximation. Developed a numerical solution of second order IVPs of Bratu-type equation using Predictor- Corrector method, among others.

Correspondence to: Nicholas S Yakusak, e-mail: *yakusaksamuel@yahoo.com*

 eISSN: 3027-060X.
How to Cite: Yakusak and Taiwo (2024). Approximate Solution of Bratu Differential Equation Using Falker – Type Method, *Scholar J,* **2(**1). https://scholarj.com/index.php/sj/article/view/13

In this paper, numerical solution of second order Initial Value Problem (IVPs) for the solution of Bratu-type equations, using Falkner-type methods of order five is investigated. The exact solution will be compared with numerical solutions.

Methodology

Derivation of the Method

In this section, we derived linear multi-step methods of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \beta_v f_{n+v} \quad (3)$$

Where α, β are unknown constants, $\alpha = 1, \beta \;/\!= 0$ and α_0, β_0 do not both vanish. We seek an approximation of the form

$$Y(x) = \sum_{j=0}^{n} \alpha_j p_j(x) \quad (4)$$

In order to obtain equation (3), $P_j(x) = x^j$ is a power series function, k is the step number, n, a_j are unknown coefficients to be determined.
Taking $k = 3$ as the step number, $v = k - r$ as the off step-point and $j = 0, 1, 2, 3$, the continuous approximation is obtained as

$$\left.\begin{aligned} y_{n+v_1} &= Y(x_{n+v_1}) \\ y'_{n+v_1} &= Y'(x_{n+v_1}) \\ y''_{n+v_1} &= Y''(x_{n+v_1}) \\ y'''_{n+1} &= Y'''(x_{n+v_1}) = f(x_{n+v_1}) \end{aligned}\right\} \quad (5)$$

Results and Discussion

Local Truncation Error and Order

The linear differential operators associated with the proposed method is of the form

$$\mathcal{L}\{y(x):h\} \equiv Y(x) - \left[\alpha_i y_{n+i} + \alpha'_i h y_{n+i} + h^2\left(\sum_{i=0}^{k} \beta_i(x) y''_{n+k} + \beta_v(x) y''_{n+v}\right)\right] \quad (6)$$

$y(x)$ is an arbitrary function.
The Taylor series expansion of (6) around x yields

$$\left.\begin{aligned} \mathcal{L}(y(x)) &= \sum_{j=0}^{r+1} C_p h^p y^p + 0(h^{r+2}) \\ C_0 &= \sum_{j=0}^{k} \alpha_j \\ C_q(-1)^q &= \left[\frac{1}{q}\sum_{j=1}^{k} j\alpha_j + \frac{1}{(q-1)!}\sum_{j=0}^{k} j^{q-1}\beta_j\right] \\ q &= 1,2,3 \ldots \end{aligned}\right\} \quad (7)$$

Definition

We say that the methods is of order $p \geq 1$, if $C_0 = C = \cdots C_p = C_{p+2} = 0$ and $C_{p+3} \neq 0$. In this case, expanding the proposed schemes in Taylor series yields the order of the method below

$$\left.\begin{aligned} C_{y_0} &= \frac{-3773}{37791360} y^{(vi)}(x_n)h^6 + 0(h^7) \\ C_{y_{\frac{1}{3}}} &= \frac{-13}{127575} y^{(vi)}(x_n)h^6 + 0(h^7) \\ C_{y_1} &= \frac{6724321}{41334300} y^{(vi)}(x_n)h^6 + 0(h^7) \\ C_{y_{\frac{4}{3}}} &= \frac{103}{16329600} y^{(vi)}(x_n)h^6 + 0(h^7) \\ C_{y_2} &= \frac{2921}{1322697600} y^{(vi)}(x_n)h^6 + 0(h^7) \\ C_{y_3} &= \frac{1371637}{13778100} y^{(vi)}(x_n)h^6 + 0(h^7) \end{aligned}\right\} \quad (8)$$

The order of the methods have order $p = 5$ (See Eq (7)).

Zero-stability and convergence

This is the concept concerning the behaviour of a numerical method with stability of the first characteristic polynomial as $h \to 0$. To analyse the zero-stability of the proposed method, the roots of the first characteristics polynomial as $h \to 0$ must be simple or less than 1.

The proposed scheme can be writtenin matrix form as

$$A^0 \tilde{Y}_\mu - A' \tilde{Y}_{\mu-1} = 0 \quad (9)$$

Where

$$\overline{Y}_\mu = (y_{n+1}, y_{n+2}, \ldots, y_{n+k})^T$$

$$\overline{Y}_{\mu-1} = (y_n, y_{n+1}, y_{n+k-r})^T$$

A^0 is the identity matrix. Following the procedure in Ramose, (2019) the proposed methods can be shown that

$$A^0 = \begin{bmatrix} 100000 \\ 010000 \\ 001000 \\ 000100 \\ 000010 \\ 000001 \end{bmatrix}, A' = \begin{bmatrix} 000010 \\ 000010 \\ 000010 \\ 000010 \\ 000010 \\ 000010 \end{bmatrix} \quad (10)$$

$$\rho(r) = |rA^0 - A'|$$

Thus, the proposed methods is zero-stable

Theorem 1. A Linear Multistep Methods (LMMs) is said to be convergent if it is consistent and zero-stable

Remarks: The roots of the proposed schemed is obtained as $\rho(r) = r^5(r-1), r = 000001$. This implies that. The proposed methods is convergent.

Region of Absolute Stability

As mentioned above, zero-stability is a concept concerning the behaviour of a numerical method for $h \to 0$. In order to know if a numerical method will give reasonable result for a given $h > 0$, we need a concept of stability different from zero-stability. Considering the stability function inform

$$M(z) = \mathfrak{h}(A - Cz - Dz^2) - B \quad (11)$$

where $z = \lambda h$ and A, B, C, E are obtained from interpolating and collocating points of the method. Computing the stability functions and its first derivative give the polynomial which can be plotted via Matlab environment. .

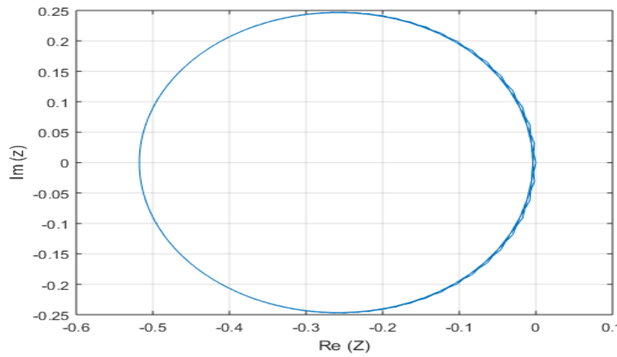

Figure 1. Region of Absolute Stability

Implementation of the Methods

In this section, the derived method will be implemented on solving some Bratus-type Differential Equations (BDEs), present our results on tables and graphs.

Example 1. Consider the Bratu-type initial value problem

$$y'' - 2e^{y} = 0, \quad y(0) = 0, \ y'(0) = 0 \quad 0 < x < 1, \ h = 0.1 \qquad (12)$$

Exact is y(x) = −2ln (cos x).

Table 1. **Numerical Solution of Example 1**

x	*Exact Solution*	*Numerical Results*	*Error*
0.1	0.010016711246470610	0.010016705039555692	6.2069 E-09
0.2	0.040269546104816700	0.040269534068873927	1.2036 E-08
0.3	0.091383311852116038	0.091383294483806199	1.7368E-08
0.4	0.16445803815011086	0.16445798257310101	5.5577E-08
0.5	0.26116848088744542	0.26116838871843090	9.2169E-08
0.6	0.38393033883887544	0.38393021221454393	1.2662E-07
0.7	0.53617151513586220	0.53617094568409092	5.6945E-07
0.8	0.72278149362268740	0.72278049873412208	9.9489E-07
0.9	1.2312529407720285	1.2312342971971592	1.8633E-05
1.0	1.581096154508463	1.5810605610750254	3.5593E-05

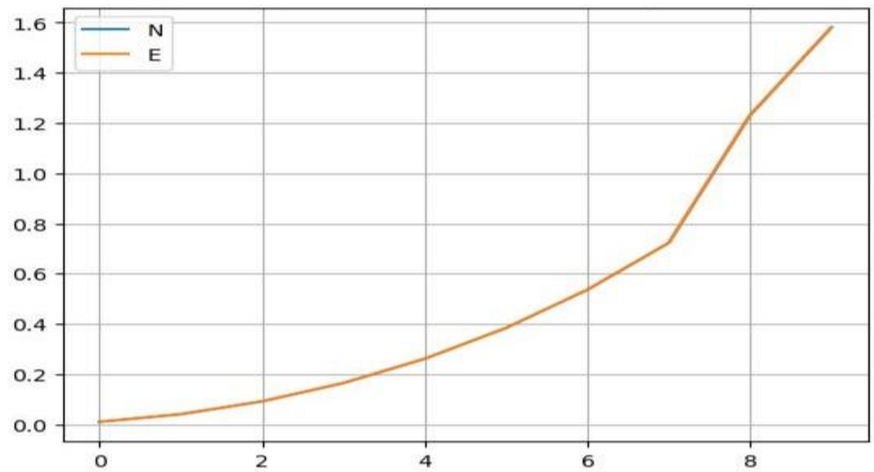

Figure 2. Graphical Presentation of Example 1

Example 2. Consider the Bratu-type initial value problem

$$y'' + \pi 2e^{-y} = 0, \quad y(0) = 0, \; y'(0) = \pi \quad 0 < x < 1, \; h = 0.1 \tag{13}$$

Exact $y(x) = \ln(\sin(\pi x))$

Table 2. Numerical Solution of Example 2

x	$Exact$ Solution	$Numerical\ Results$	$Error$
0.1	0.26936833397496114	0.26937161481672322	3.28084E-06
0.2	0.46246895987938497	0.46247558674988345	6.6268E-06
0.3	0.59290681818485150	0.59291697911335306	1.0161E-06
0.4	0.66845107369154235	0.66846508645442245	1.4013E-05
0.5	0.69314708062662442	0.69316564518558186	1.85646E-5
0.6	0.66825071630680667	0.66827473800143076	2.4022E-05
0.7	0.59249578392011496	0.59252733482295458	3.1551E-05
0.8	0.46182436825450018	0.46186511634825410	4.0748E-05
0.9	0.26844914001894383	0.26850153969447523	5.2399E-05

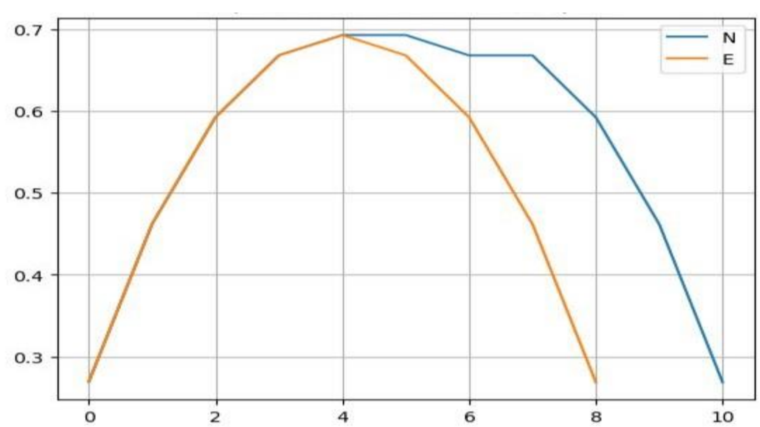

Figure 3. Graphical Presentation of Example 2

Conclusion

In this study, the method for finding an approximate solution of Bratu's Differential Equations (BDEs) using Falkner-type method of order five is proposed. All the numerical results implemented, shows that we have favourably applied Falkner-type method to obtain approximate solution of the BDEs. The proposed method have good region of stability and converges. The obtained results are very close to the exact solutions, this indicate that a little iteration of the method will result in some useful results. As the result seems necessary from the authors' point of view, the suggested technique has the potentials to be practical in solving other similar Ordinary Differential Equations (ODEs).

References

[1] Abukhaled, M., Khuri, S. and Sayfy, A. (2012) "Spline-based numerical treatments of bratu-type equations," *Palestine J. Math,* vol. 1, pp. 63–70, 2012

[2] Agheli, B. (2021). Approximate Solution Of Bratu Differential Equations Using Trigonometric Basic Functions. *Kragujevac Journal of Mathematics*. 45(2): 203–214

[3] Batiha, B. (2010) "Numerical solution of bratu-type equations by the variational iteration method," Hacettepe. *Journal of Mathematics and Statistics,* vol. 39, no. 1.

[4] Christodoulou, N. (2009) "An algorithm using runge–kutta methods of orders 4 and 5 for systems of odes," *International journal of numerical methods and applications,* vol. 2, no. 1, pp. 47–57.

[5] Feng, X., He, Y. and Meng, J. (2008) "Application of homotopy perturbation method to the bratu-type equations," Topological Methods in Nonlinear Analysis, vol. 31, no. 2, pp. 243–252.

[6] Fateme, G. and Shateyi, S. (2019). A new approach for solving Bratu's problem. *Demonstr. Math.* 52:336–346

[7] Fenta, H. B. and Derese, G. H., (2019). Numerical Solution of Second Order Initial Value Problems of Bratu-type Equations using Sixth Order Runge-Kutta Seven Stages Method. International Journal of Computing Science And Applied Mathematics, VOL. 5, NO. 1

[8] Wazwaz, A.M., (2005) Adomian decomposition method for a reliable treatment of the Bratu-type equations. *Appl. Math. Comput*. 166:652–663.